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THE BLOW-UP SOLUTIONS OF THE HEAT DISSIPATION EQUATION WITH NONLINEAR BOUNDARY FLUX

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Abstract
. In this paper, we investigate the boundary problem for the doubly nonlinear heat dissipation equation. We also use the self-similar solution construction method to solve the problem. Global or blow-up theorems of the solution are proved.

Keywords. Blow-up, global, density, self-similar solution, mathematical modeling.

Introduction

Let the heat dissipation equation with doubly nonlinear terms in the domain $Q=\{(t,x):\ 0<t\le T,\ x\in R_+^1\}$ be given

$$\rho(x)u_t=(|u_x^m|^{(p-2)}u_x^m)_x \quad (x,t)\in R_+\times(0,+\infty) \tag{1}$$

Here, $\rho(x)$ is a density function defined as follows:

$\rho(x)=(1+|x|)^{-n}$ and $m,p>2,n$ are constant positive parameters. For equation (1), the following boundary flux condition with nonlinearity is given:

$$-|u_x^m|^{(p-2)}u_x^m(0,t)=u^q(0,t),t>0 \tag{2}$$

and the initial condition is given as:

$$u(x,0)=u_0(x)\ge 0, x\in R_+ \tag{3}$$

conditions are provided. The Cauchy problem for the doubly nonlinear equation is fundamental in the mathematical modeling of various processes, including heat conduction, fluid and gas filtration, and diffusion phenomena in media with doubly nonlinear coefficients and variable density.

Equation (1) encompasses both slow and fast diffusion as well as the p-Laplace equations. If $u=0$ or $\nabla u=0$ in a certain region, equation (1) exhibits degeneracy, meaning the type of the equation changes. In such regions, the derivatives of the solution may have discontinuities, and equation (1) might not admit a classical solution [1]. Therefore, a weak (generalized) solution is sought and studied for equation (1).

Research methodology

Authors of the work [3] established asymptotic of the solutions and gave global solvability for the following problem

$$\rho(x)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right),\ (x,t)\in R_+\times(0,+\infty),$$

$$u(0,x)=u_0(x),\ x\in R^N.$$

In [4], the characteristics of the reaction-diffusion system with two types of inefficiencies are studied. In the study, it was determined how numerical parameters affect the development of the process. The existence of bounded and "fading" solutions is proved and their long-time behavior is described. Also, the conditions for finding a general solution of the Cauchy problem are defined and a special estimate (of the Knerr-Kersner type) for free boundaries is defined.

$$Au\equiv-\rho(x)\frac{\partial u}{\partial t}+L(n,m,p)u+\varepsilon\gamma(t)\rho(x)u^{\beta}=0,$$

$$u(0,x)=u_0(x),\ x\in R^N.$$

where $L(n,m,p)u=\nabla\left(\left|x^n\right|u^{m-1}\left|\nabla u\right|^{p-2}\nabla u\right),\ \beta\geq1,n,p,m$ -are the given numerical parameters, $\nabla(\cdot)-grad_x(\cdot),0<\gamma(t)\in C(0,\infty),\varepsilon=\pm1,\rho(x)=|x|^{\delta-2}$.

In [5], In this article, as part of the research of new properties of mathematical models representing nonlinear diffusion, filtration, and heat dissipation processes, the analysis of heat dissipation in the medium with exponential density and in the presence of absorption is carried out.

$$\rho(x)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right),\ t>0,x\in R^1$$

and

$$u(x,0)=u_0(x)\geq0$$

Where $\rho(x)=e^{2x}>0$ density(exponential) function. Furthermore,

$$\rho(x)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\frac{\partial u}{\partial x}\right)-\rho(x)\cdot\gamma u,\ t>0,x\in R^1,\gamma>0$$

problem is studied.

In [6], the properties of self-similar solutions of the Cauchy problem for the equation with double nonlinear exponential effects are studied. The existence conditions of Fujita-type global solutions are defined. Also, the conditions for having a lower solution for the equation were found.

$$\rho(x)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right),\ (t,x)\in Q,$$

and

$$u(t,x)|_{t=0}=u_0(x)\geq0,\ x\in R,$$

where $Q=\{(t,x):t>0, x\in R\}$, $p\geq 2, \sigma \in R_+, \rho(x)=e^x$. In the case $\sigma+p-2>0$, it is shown that there exist the solutions $u(t,x)$, which possess the property of the finite velocity of the propagation of perturbation. It means that for every $t>0$ there exists such a continuous function $l(t)$, that $u(t,x)\equiv 0$ as $|x|\geq l(t)$ (in the linear case when $\sigma=0,\ p=2$ it is trivial). The surface $|x|=l(t)$ is called a front of perturbation or a free boundary. The solution of the equation (1) satisfying $l(\infty)<+\infty$, and $u(t,x)\equiv 0$ for $|x|\geq l(t)$ is called a localized solution.

Sources [7-11] consider the Cauchy problem for the heat equation with variable density and two inadequacies. Since these equations are reduced to first-order equations, the Cauchy problem usually does not have a classical solution. Therefore, generalized (weak) solutions satisfying the equation in the sense of distributions are studied.

Self-similar solutions for the equation are constructed using standard equations and arbitrary separation methods. A theorem proving the existence of a global solution for small initial data of the problem is presented. To prove this, the upper solution is constructed at certain parameter values. The asymptotic behavior of the super solution with limited support is defined. Also, the conditions for the environment parameters for the case where the self-similar solution is asymptotic have been found.

In the given sphere $Q=\{(t,x)\,|\,t>t_0>0, x\in R\}$, let us consider the Cauchy problem for the double nonlinearity parabolic equation

$$Au\equiv -\rho(x)\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(u^{\sigma}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+\rho(x)u^{\beta}=0$$

$$u\big|_{t=t_0}=u_0(x)\geq 0,\ x\in R$$

considering initial data that is neither trivial or negative. Where, $\sigma>0, p\geq 2, \beta\geq 1$ are the given parameters, $\rho(x)=e^{\alpha x}$.

Construction of solution

In this article, we examine the conditions under which the solutions of problem (1)-(3) are global or blow-up behavior.

Theorem 1. If $q\leq\dfrac{m(p-1)(2-n)+n-1}{(p-1)m+1}$ holds, then every solution of the problem (1)-(3) will be global.

Proof. To find the upper solution for the equation (1), we consider the following model of the solution:

$$\bar{u}(x,t)=e^{Lt}(K+e^{-M\xi})^{(1/m)},\ \ \xi=(1+x)e^{(-Jt)}$$

Where

$$K>\|u_0\|_{\infty}, L=\frac{M^{p}(p-1)}{K^{\frac{1}{m}}}, M=(K+1)^{\frac{q}{m(p-1)}}, J=((p-1)m-1)/((p-1)m+1)L$$

After the calculations, we obtain the following results:

$$\rho(x)\frac{\partial \bar{u}}{\partial t}=\xi^{(-n)}e^{(L-nJ)t}[L(K+e^{-M\xi}))^{(1/m)}-M/me^{-M\xi})(-J\xi)(K+e)^{(-M\xi)((1/m-1)}]\geq$$
$$L\xi^{(-n)}e^{(L-nJ)t}(K+e^{-M\xi})^{(1/m)}\geq$$
$$L\xi^{(-n)}e^{(L-nJ)t}K^{(1/m)}$$
$$\left|\frac{\partial \bar{u}^m}{\partial x}\right|^{p-2}\frac{\partial \bar{u}^m}{\partial x}=e^{mL(p-1)t}\left|-Me^{-M\xi}\right|^{(p-2)}\left(\left|-Me^{-M\xi}\right|\right)e^{-J(p-1)t}=-M^{(p-1)}e^{(mL-J)(p-1)t}e^{-M(p-1)\xi}$$
$$\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}^m}{\partial x}\right|^{p-2}\frac{\partial \bar{u}^m}{\partial x}\right)=M^p(p-1)e^{(mL(p-1)-Jp)t}e^{-M(p-1)\xi}$$

We will show that the function $\bar{u}(x,t)$ is an upper solution to the problem (1)-(3). According to the comparison principle, it must satisfy the following inequality:

$$\rho(x)\frac{\partial \bar{u}}{\partial t}\geq\frac{\partial}{\partial x}\left(\left|\frac{\partial \bar{u}^m}{\partial x}\right|^{p-2}\frac{\partial \bar{u}^m}{\partial x}\right),\ (x,t)\in R_+\times(0,+\infty) \tag{4}$$

$$-|\bar{u}_x^m|^{(p-2)}\ \bar{u}_x^m(0,t)=\bar{u}^q(0,t),t>0 \tag{5}$$

For the satisfaction of inequalities (4)-(5), we introduce conditions using the sought upper solution in the model form and perform the necessary calculations. According to inequality (4), we have:

$$L\xi^{(-n)}e^{(L-nJ)t}K^{(1/m)}\geq M^p(p-1)e^{(mL(p-1)-Jp)t}e^{-M(p-1)\xi}$$

and according to inequality (5), we obtain

$$\left.\left|\frac{\partial \bar{u}^m}{\partial x}\right|^{p-2}\frac{\partial \bar{u}^m}{\partial x}\right|_{x=0}=-M^{(p-1)}e^{(mL-J)(p-1)t}e^{-M(p-1)\xi}\ |_{x=0}\geq e^{qLt}(K+e^{-M\xi_0})^{(q/m)}$$

and from this, we introduce the following conditions:

$$\begin{cases}L-nJ\geq mL(p-1)-Jp\\(mL-J)(p-1)\geq qL\end{cases}$$

and we obtain $q\leq 1+\dfrac{J(1-n)}{L}$.

Additionally, by introducing the conditions $M^{(p-1)}=(K+1)^{(q/m)}$, $K^{(1/m)}\geq M^p(p-1)$, we obtain $J=((p-1)m-1)/((p-1)m+1)L, M=(K+1)^{(q/m(}p-1))$, and through J,M,L,K, we derive $q\leq 1+(1-n)\dfrac{(p-1)m-1}{(p-1)m+1}=\dfrac{m(p-1)(2-n)+n-1}{(p-1)m+1}$, ensuring that inequalities (4)-(5) hold. Therefore, $\bar{u}(x,0)\geq u_0(x)$ and $\bar{u}(0,0)\geq u_0(0)$ are satisfied. As a result, by the comparison theorem, Theorem 1 is proved.

Theorem 2. If $q\geq\dfrac{(p-1)(m(n-1)+1)}{n-1}$, then every solution of the problem (1)-(3) will exhibit a blow-up behavior.

Proof. We seek a blow-up solution in the following model form:

$$u(x,t)=(T-t)^{\alpha}\varphi(\eta),\eta=(1+x)(T-t)^{(-\beta)} \tag{6}$$

Here, $\alpha=\dfrac{1-p}{(p-1)[m(n-1)-1]+q(p-n)}$, $\beta=\dfrac{q-m(p-1)}{(p-1)[m(n-1)-1]+q(p-n)}$ and $\varphi(\eta)$ are the solutions of the problem under consideration. In this case, we perform the necessary calculations.

$1+x=\eta(T+t)\beta,\eta_t=-\beta\eta/(T+t),\eta_x=(T+t)^{(-\beta)}$

$\rho(x)\partial u/\partial t=\eta^{(-n)}(T+t)^{(-\beta n+\alpha-1)}[\alpha\varphi-\beta\eta\varphi_{\eta}]=(T+t)^{\alpha m(p-1)-\beta p}(|\varphi_{\eta}^{m}|^{(p-2)}\varphi_{\eta}^{m})_{\eta}$

$$\begin{cases}\alpha m(p-1)-\beta p=-\beta n+\alpha-1\\ \alpha m(p-1)-\beta(p-1)=\alpha q\end{cases} \tag{7}$$

The solution of the system (7) allows us to find the coefficients α and β, which are strong and equal to the following:

$$\begin{cases}\alpha(m(p-1)-q)+\beta(1-p)=0\\ \alpha(m(p-1)-1)+\beta(n-p)=-1\end{cases}$$

We find these values and, from this, derive the following for the coefficients α and β:

$$\alpha=\frac{1-p}{(p-1)[m(n-1)-1]+q(p-n)},\ \beta=\frac{q-m(p-1)}{(p-1)[m(n-1)-1]+q(p-n)}$$

Taking into account the conditions (7), equation (6) transforms into the following model equation:

$$(\varphi_{\eta}^{m}|^{(p-2)}\varphi_{\eta}^{m}|)_{\eta}+\beta\eta^{1-n}\varphi_{\eta}-\alpha\eta^{-n}\varphi=0 \tag{8}$$

$$-|\varphi_{\eta}^{m}|^{(p-2)}\varphi_{\eta}^{m}|_{\eta=\eta_0}=\varphi^{q}(\eta_0) \tag{9}$$

Here, we seek the function ϕ(η)in the following form

$\varphi(\eta)=E(a-\eta^{\gamma_1})_{+}^{\gamma_2}$

We perform the necessary calculations and after the calculations, we introduce the following conditions:

$$\gamma_1-n=(\gamma_1-1)p=\gamma_1 p-p,\gamma_2-1=\gamma_2 m(p-1)-p$$

And from this, we find the value of the $\gamma_1=\dfrac{p-n}{p-1},\gamma_2=\dfrac{p-1}{m(p-1)-1}$ parameter, which can be used to determine the parameter. To do this, we perform the following calculations and introduce the conditions:

$E^{(p-2)}=\beta m^{(1-p)}(\gamma_1\gamma_2)^{(1-p)}.$

As a result, we obtain $E=(\beta m^{(1-p)}(\gamma_1\gamma_2)^{(1-p)})^{\frac{1}{p-2}}$.

Conclusion

In this work, we have analyzed the blow-up and global solutions for the heat dissipation equation with nonlinear boundary flux and variable density. Through the application of mathematical techniques such as the comparison principle and weak solutions, we have derived conditions under which the solutions exhibit either global behavior or blow-up phenomena.

We showed that, under certain conditions (such as $q > 1$), blow-up behavior is inevitable, while for other parameter values, global solutions exist. The results are significant for understanding the complex dynamics of diffusion processes in media with nonlinearities and variable coefficients, which arise in various physical, chemical, and biological contexts.

By employing model solutions and conducting necessary calculations, we derived key inequalities and conditions that govern the behavior of the system. These findings contribute to the broader understanding of nonlinear diffusion equations and their applications in real-world processes.

In conclusion, the results provide a deeper insight into the mathematical modeling of diffusion processes with nonlinear characteristics and serve as a foundation for further research in this area.

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