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TWO AND THREE PARTICLE BRANCHES OF THE ESSENTIAL SPECTRUM OF A 3×3 OPERATOR MATRICES

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ABSTRACT
In the present paper, we precisely describe the location and structure of the essential spectrum of a 3×3 operator matrix A_μ,μ>0 associated to a system describing three particles in interaction, without conservation of the number of particles, in the quasi-momentum representation. Two and three-particle branches of the essential spectrum of A_μ are identified. The number of segments of the essential of A_μ is studied with respect to the parameter μ>0.

Keywords: operator matrix, essential spectrum, branches.

In statistical physics [1, 2], solid-state physics [3] and the theory of quantum fields [4] some important problems arise where the number of quasi-particles is bounded, but not fixed. In [5] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number. Recall that the study of systems describing N $(1 \leq N < \infty)$ particles in interaction, without conservation of the number of the particles, is reduced to the investigation of the spectral properties of self-adjoint operators, acting in the *cut subspace* $\mathcal{H}^{(N)}$ of Fock space, consisting of $n \leq N$ particles [2-5]. We note that the location and structure of the model operators acting in $\mathcal{H}^{(3)}$ are studied in detail in [6-8].

Denote by $\mathbb{T}^1$ the one-dimensional torus. Let $\mathcal{H}_0 := \mathbb{C}$ be the field of complex numbers, $\mathcal{H}_1 := L_2(\mathbb{T}^1)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^1$ and $\mathcal{H}_2 := L_2^s(\mathbb{T}^2)$ be the Hilbert space of square-integrable symmetric (complex) functions on $\mathbb{T}^2$.

Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_0, \mathcal{H}_1$ and $\mathcal{H}_2$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. By the definition a block operator matrix is a matrix the entries of which are linear operators and every bounded linear operator acting in the Hilbert spaces $\mathcal{H}$ can be written as a block operator matrix of order 3.

In the present paper we consider the operator $\mathcal{A}_\mu, \mu > 0$ acting in the Hilbert space $\mathcal{H}$ as a block operator matrix

$$\mathcal{A}_\mu = \begin{pmatrix} A_{00} & \mu A_{01} & 0 \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ 0 & \mu A_{12}^* & A_{22} \end{pmatrix}, \quad (1)$$

with the matrix elements $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i, i \leq j, i, j = 0,1,2$ are defined by

$$A_{00}f_0 = \varepsilon f_0, \qquad A_{01}f_1 = \int_{\mathbb{T}^1} \sin 3t\, f_1(t)\, dt$$

$$(A_{11}f_1)(x) = (\varepsilon + 1 - (\cos 3x))f_1(x),$$

$$A_{12}f_2(x) = \int_{\mathbb{T}^1} \sin 3t\, f_2(x,t)dt$$

$$(A_{22}f_2)(x,y) = (\varepsilon + 2 - (\cos 3x) - (\cos 3y))f_2(x,y).$$

Here $f_i \in \mathcal{H}_i, i = 0,1,2$. Under these assumptions the operator matrix $\mathcal{A}_\mu$ defined by the formula (1) is linear, bounded and self-adjoint in $\mathcal{H}$.

We remark that the operators A_{01}, A_{12} and A_{01}^*, A_{12}^* are called annihilation and creation operators [4], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

It is clear that

$$(A_{01}^* f_0)(x) = sin3x \cdot f_0, \quad f_0 \in \mathcal{H}_0;$$

$$(A_{12}^* f_1)(x,y) = \frac{1}{2}\big(sin3x \cdot f_1(y) + sin3y \cdot f_1(x)\big), \qquad f_1 \in \mathcal{H}_1.$$

In order to study the essential and discrete spectra of the operator matrix $\mathcal{A}_\mu$ we introduce a generalized Friedrichs model $h_\mu, \mu > 0$ which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$h_\mu := \begin{pmatrix} A_{00} & \mu A_{01} \\ \mu A_{01}^* & A_{11} \end{pmatrix}.$$

It's matrix elements $A_{ij}, i \le j, i, j = 0,1$ are given in the above. It is not difficult to prove that operator h_μ is linear, bounded and self-adjoint. We consider operator matrix h_0 in the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$h_0 := \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}$$

The perturbation $h - h_0$ of the operator h_0 is a bounded self-adjoint operator matrix of rank 2. From the definition one can conclude that the spectrum of h_0 is equal to

$$\sigma(h_0) = \sigma(A_{00}) \cup \sigma(A_{11})$$

where

$$\sigma(A_{00}) = \sigma_{\rm disc}(A_{00}) = \{\varepsilon\}; \qquad \sigma(A_{11}) = \sigma_{\rm ess}(A_{11}) = [\varepsilon; \varepsilon + 2].$$

According to the famous Weyl's theorem on the conservation of the essential spectrum under finite rank perturbations implies that the essential spectra of operators h_0 and h_μ coincide. Therefore

$$\sigma_{\rm ess}(h_\mu) = \sigma_{\rm ess}(h_0) = [\varepsilon, \varepsilon + 2].$$

One can see that

$$\min_{x \in \mathbb{T}^1}(\varepsilon + 1 - (\cos 3x)) = \varepsilon + 1 - 1 = \varepsilon$$

$$\max_{x \in \mathbb{T}^1}(\varepsilon + 1 - (\cos 3x)) = \varepsilon + 1 + 1 = \varepsilon + 2$$

We consider the eigenvalue equation $h_\mu f = zf$. This equation can be written as the following system of equation:

$$\begin{cases} \varepsilon f_0 + \mu \int_{\mathbb{T}^1} sin(3t) f_1(t) dt = z f_0 \\ \mu \sin(3x) f_0 + (\varepsilon + 1 + \cos(3x)) f_1(x) = z f_1(x) \end{cases} \quad (1)$$

We write the second equation of the system of equation (1) in the form:

$$\mu \sin(3x) f_0 = (z - \varepsilon - 1 + \cos(3x)) f_1(x) \quad (2)$$

Since $z \notin [\varepsilon; \varepsilon + 2]$, for any $x \in \mathbb{T}^1$ we have $\varepsilon + 1 - \cos(3x) - z \neq 0$. Then from the equality (2) for $f_1(x)$ we obtain

$$f_1(x) = \frac{\mu \sin(3x)}{z - \varepsilon - 1 + \cos(3x)} f_0 \quad (3)$$

Substituting the expression (3) for $f_1(x)$ into the first equation in the system (1), we obtain

$$\varepsilon f_0 + \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)\, dt}{z - \varepsilon - 1 + \cos(3t)} f_0 - z f_0 = 0$$

or

$$f_0 \left(\varepsilon - z + \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t) dt}{\varepsilon - z + 1 - \cos(3t)} \right) = 0$$

For any fixed $\mu > 0$, we define an analytic function $\Delta_\mu(\cdot)$ in $\mathbb{C} \backslash [\varepsilon; \varepsilon + 2]$ by:

$$\Delta_\mu(z) := \varepsilon - z - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t) dt}{\varepsilon + 1 - \cos(3t) - z}.$$

Usually the function $\Delta_\mu(\cdot)$ is called the Fredholm determinant associated with the operator matrix h_μ. Let us establish a relation between the eigenvalues of h_μ and zeros of $\Delta_\mu(\cdot)$.

Lemma 1. *For any fixed* $\mu > 0$ *the operator matrix* $\mathcal{A}_\mu$ *has the eigenvalue* $z_\mu \in \mathbb{C} \backslash [\varepsilon; \varepsilon + 2]$ *if and only if* $\Delta_\mu(z_\mu) = 0$.

From Lemma 1 it follows that for the discrete spectrum of $\mathcal{A}_\mu$ the equality

$$\sigma_{\text{disc}}(\mathcal{A}_\mu) = \{z \in \mathbb{C}[\varepsilon; \varepsilon + 2] \colon \Delta_\mu(z) = 0\}$$

holds.

The following lemma describes the number and location of the operator matrix $\mathcal{A}_\mu$.

Lemma 2. *For any* $\mu > 0$ *the operator matrix* h_μ *has two eigenvalues, one to the left of* ε *and the other to the right of* $\varepsilon + 2$.

For the spectrum of operator matrix h_μ we have

$$\sigma(h_\mu) = \{E_\mu^{(1)}\} \cup [\varepsilon; \varepsilon + 2], \quad \text{if} \quad \mu \leq {}^1\!/\!_{\sqrt{\pi}},$$

$$\sigma(h_\mu) = \{E_\mu^{(1)}\} \cup [\varepsilon; \varepsilon + 2] \cup \{E_\mu^{(2)}\}, \quad \text{if} \quad \mu > {}^1\!/\!_{\sqrt{\pi}}$$

where the numbers $E_\mu^{(1)}$ and $E_\mu^{(2)}$ are zeros of $\Delta_\mu(\cdot)$ where

$$E_\mu^{(1)} < \varepsilon, \quad E_\mu^{(2)} > \varepsilon + 2.$$

Indeed,

$$\Delta_\mu(\varepsilon) = \varepsilon - \varepsilon - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{\varepsilon + 1 - \cos(3t) - \varepsilon} = -\mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{1 - \cos(3t)} =$$

$$= -\mu^2 \int_{\mathbb{T}^1} \frac{(1 - \cos(3t))(1 + \cos(3t))}{1 - \cos(3t)} dt = -\mu^2 \int_{\mathbb{T}^1} dt = -2\pi\mu^2 < 0;$$

$$\Delta_\mu(\varepsilon + 2) = \varepsilon - \varepsilon - 2 - \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{\varepsilon + 1 - \cos(3t) - \varepsilon - 2} =$$

$$= -2 + \mu^2 \int_{\mathbb{T}^1} \frac{\sin^2(3t)dt}{1 + \cos(3t)} = -2 + \mu^2 \int_{\mathbb{T}^1} \frac{(1 - \cos(3t))(1 + \cos(3t))\, dt}{1 + \cos(3t)} =$$

$$= -2 + \mu^2 \int_{\mathbb{T}^1} (1 - \cos(3t))dt = -2 + \mu^2 \int_{\mathbb{T}^1} dt = -2 + \mu^2 2\pi;$$

From here one can see that

$$\Delta_\mu(\varepsilon + 2) > 0, \quad \text{if} \quad \mu > {}^1\!/_{\sqrt{\pi}};$$
$$\Delta_\mu(\varepsilon + 2) \le 0, \quad \text{if} \quad \mu \le {}^1\!/_{\sqrt{\pi}}.$$

The following theorem describes the location of essential spectrum of $\mathcal{A}_\mu$.

Theorem 1. *The essential spectrum* $\sigma_{\text{ess}}(\mathcal{A}_\mu)$ *of* $\mathcal{A}_\mu$ *satisfies*

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = \left[E_\mu^{(1)} + \varepsilon;\ E_\mu^{(1)} + \varepsilon + 2\right] \cup [\varepsilon; \varepsilon + 4], \quad \text{if} \quad \mu \le {}^1\!/_{\sqrt{\pi}};$$

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = \left[E_\mu^{(1)} + \varepsilon;\ E_\mu^{(1)} + \varepsilon + 2\right] \cup [\varepsilon; \varepsilon + 4] \cup \left[E_\mu^{(2)} + \varepsilon;\ E_\mu^{(2)} + \varepsilon + 4\right],$$
$$\text{if} \quad \mu > {}^1\!/_{\sqrt{\pi}}.$$

Definition 1. The sets $\sigma_{\text{two}}(\mathcal{A}_\mu)$ and $\sigma_{\text{three}}(\mathcal{A}_\mu) = [\varepsilon; \varepsilon + 4]$ are called the two-particle and three-particle branches of the essential spectrum $\sigma_{\text{ess}}(\mathcal{A}_\mu)$ of $\mathcal{A}_\mu$, where

$$\sigma_{\text{two}}(\mathcal{A}_\mu) = \left[E_\mu^{(1)} + \varepsilon;\ E_\mu^{(1)} + \varepsilon + 2\right] \cup [\varepsilon; \varepsilon + 4], \quad \text{if} \quad \mu \le {}^1\!/_{\sqrt{\pi}};$$

$$\sigma_{\text{two}}(\mathcal{A}_\mu) = \left[E_\mu^{(1)} + \varepsilon;\ E_\mu^{(1)} + \varepsilon + 2\right] \cup [\varepsilon; \varepsilon + 4] \cup \left[E_\mu^{(2)} + \varepsilon;\ E_\mu^{(2)} + \varepsilon + 4\right],$$

if $\mu > {}^1\!/_{\sqrt{\pi}}$

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